Lower Bounds for DNF-Refutations of a Relativized Weak PHP

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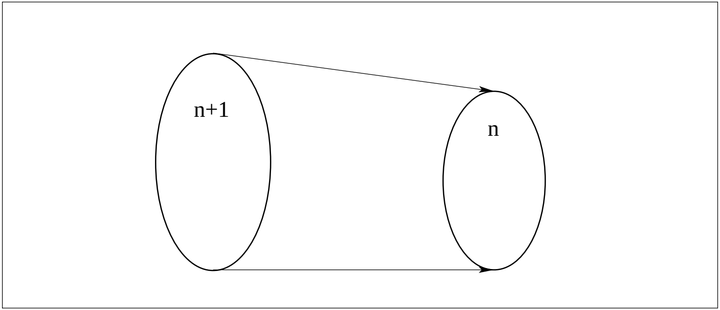
Part I

BACKGROUND

Pigeonhole Principle

PHP$_n^{n+1}$:

If $n+1$ pigeons fly to n holes, then some hole gets overbooked.



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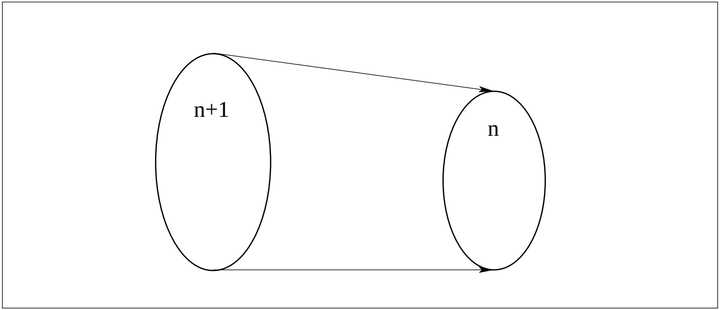

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Weak Pigeonhole Principle

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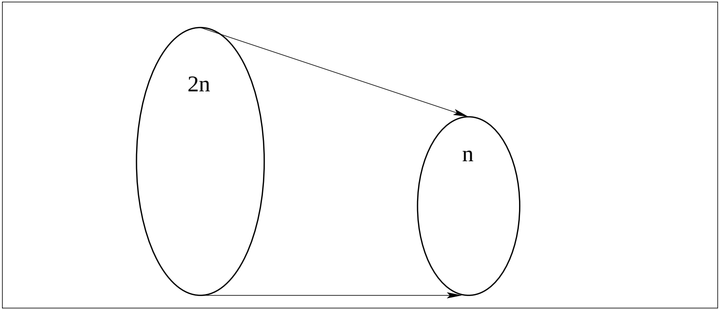

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Poly-size AC^0 circuits can approximately count [AB84, A93].

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However:

Does PHP_n^{2n} have poly-size AC^0-refutations? OPEN

What's Known?

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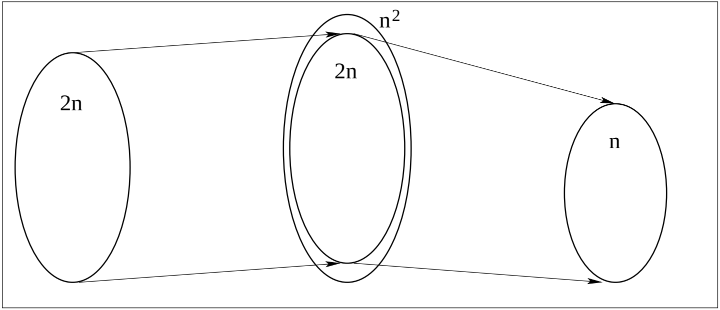
4. PHP_n^{2n} doesn't have subexp-size $\Sigma_{2,(\log n)^{1-\epsilon}}$-proofs [R02].

A Relativized PHP

$\mathbf{PHP}^{\ell,m}_n$ with $\ell > m > n$:

If can't be that m out of ℓ pigeons fly but only n make it.



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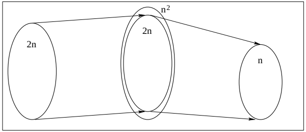
Upper bound and equivalence:

F1: $\mathrm{PHP}_n^{n^2,2n}$ has quasipoly-size proofs in $\Sigma_{2,\mathrm{polylog}}$.
F2: $\mathrm{PHP}_n^{n^2,2n} \equiv \mathrm{PHP}_n^{2n}$ mod poly-size AC^0-proofs.

The Formula: $\mathrm{PHP}_n^{n^2,2n}$



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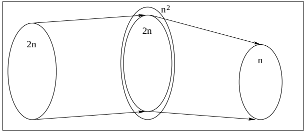

Variables:

- $x_{i,j}$: the i-th flying pigeon is j.
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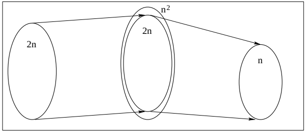

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Clauses:

- some pigeon is pigeon number i,
- pigeon j cannot be both the i-th and the i'-th, for $i \neq i'$,
- if pigeon j is the i-th, then j flies and it reaches some hole,
- if j and j' fly, then they do not both reach hole k, for $j \neq j'$.

Part II

MAIN RESULT

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Theorem:

Every Σ_2-proof of $PHP_n^{n^2,2n}$ has size $n^{(\log n)^{1/2-o(1)}}$.

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Reaching depth-2 without hiding Ajtai's method in it:

first super-polynomial lower bound for $\Sigma_{>1}$-proofs that does not follow from Ajtai's forcing method.

Part III

PROOF SKETCH

Random Restriction Method in Proof Complexity

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Discouraging obstacle:

there is little hope this technique would yield superpolynomial, but not superquasipolynomial, lower bounds

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An obstacle:

the length in step 5 is "number of mentioned pigeons"
instead of number of variables as we would need

Analysis of step 5

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$$\Pr[\ t \text{ survives }] \leq \left(\frac{2n}{n^2 - s} + \frac{1}{2} \right)^s$$

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L2: If t mentions less than s pigeons and $k < s$, then

$$\Pr[\ t \text{ mentions } \geq k \text{ pigeons }] \leq \binom{s}{k}\left(\frac{2n}{n^2}\right)^k$$

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Obstacle:

resilient expanders require degree $\geq \Omega(\log n / \log \log n)$
so we cannot afford more than $k = o(1)$.

Bypassing the obstacle: second attempt

Idea:

restrict to resilient expander but encode flights in binary!

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Solvable technicalities:

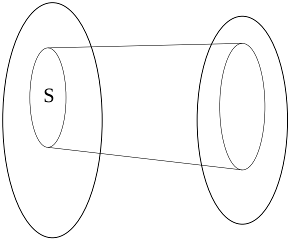
1. depth increases by 1 (solution: not really),
2. formula changes (solution: reprove height lower bound for it).

Part IV

RESILIENT EXPANDERS

Bipartite expanders

(s, e)-**expander**: $|N_G(S)| \geq (1 + e)|S|$



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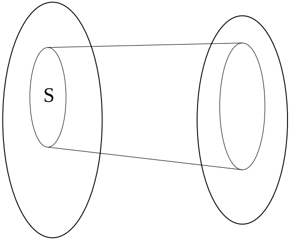
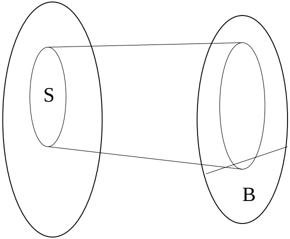


(s, e, b)-**resilient expander**: $\Pr_B \left[\, |N_{G \setminus B}(S)| \geq (1 + e)|S| \,\right] \geq 1/2$



Parameters

Fix $|L| = 2n$, $|R| = n$, $s = n^{1-\Omega(1)}$, $e = 1$, and $b = \Theta(n/\log n)$.

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for what d_L's is G an (s, e, b)-resilient expander?

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$$\Omega\left(\frac{\log n}{\log\log n}\right) \leq d_L \leq O(\log n)$$

Interesting part of lemma

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Proof of (1): Assume $d_L = o(\log n / \log\log n)$.

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Proof of (1): Assume $d_L = o(\log n / \log\log n)$.

$\Rightarrow$ also $d_R = O(\log n)$ whp,

$\Rightarrow$ there are $\Omega(n/(\log n)^2)$ vertices with disjoint neighborhoods,

Interesting part of lemma

Lemma:

$$\Omega\left(\frac{\log n}{\log\log n}\right) \overset{(1)}{\leq} d_L \overset{(2)}{\leq} O(\log n)$$

Proof of (1): Assume $d_L = o(\log n/\log\log n)$.

$\Rightarrow$ also $d_R = O(\log n)$ whp,

$\Rightarrow$ there are $\Omega(n/(\log n)^2)$ vertices with disjoint neighborhoods,

$\Rightarrow$ probability that every L-vertex has some neighbor outside B is

$$\leq \left(1 - \left(\frac{1}{\Theta(\log n)}\right)^{o(\log n/\log\log n)}\right)^{n/(\log n)^2} = o(1).$$

Q.E.D.

(thanks)